

ON AUTOMATIC HOMEOMORPHICITY FOR TRANSFORMATION MONOIDS

CHRISTIAN PECH AND MAJA PECH

ABSTRACT. Transformation monoids carry a canonical topology — the topology of point-wise convergence. A closed transformation monoid \mathfrak{M} is said to have automatic homeomorphicity with respect to a class \mathcal{K} of structures, if every monoid-isomorphism of \mathfrak{M} to the endomorphism monoid of a member of \mathcal{K} is automatically a homeomorphism. In this paper we show automatic homeomorphicity-properties for the monoid of non-decreasing functions on the rationals, the monoid of non-expansive functions on the Urysohn space and the endomorphism-monoid of the countable universal homogeneous poset.

A major question in mathematics is to what extent the symmetries of a structure determine its properties (algebraic, geometric,...). Of course the answer to this question depends strongly on the decision, what we consider to be a structure. For geometries this is essentially Felix Klein’s Erlangen Program. However, if we take a much broader point of view and consider model theoretic structures, then the answer is that the automorphism group in general says little about the properties of a structure. Indeed, in some sense “most” structures have no nontrivial symmetries.

This situation changes if we restrict the class of structures in question. For instance, by the Ryll-Nardzewski Theorem, a countable structure is ω -categorical (i.e., determined among other countable structures up to isomorphism by its elementary theory) if and only if its automorphism group is oligomorphic (i.e., it has finitely many k -orbits for every $k \in \mathbb{N} \setminus \{0\}$). More or less a direct consequence of this theorem is that if \mathbf{A} is a countable ω -categorical structure and if \mathbf{B} is a countable structure (possibly of different type than \mathbf{A}), then $\text{Aut}(\mathbf{A})$ and $\text{Aut}(\mathbf{B})$ are isomorphic as permutation groups if and only if \mathbf{A} and \mathbf{B} are first-order interdefinable.

Every permutation group can be endowed with a natural topology — the topology of pointwise convergence—under which the group operations are continuous. It was shown by Coquand, Ahlbrandt and Ziegler

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(cf. [1]) that two countable ω -categorical structures are first order bi-interpretable if and only if their automorphism groups are isomorphic as topological groups.

In [10] it was shown by Dixon, Neumann and Thomas that in the full symmetric group $\mathfrak{S}_{\mathbb{N}}$ on \mathbb{N} every subgroup of index less than 2^{\aleph_0} is open. In general, a countable structure \mathbf{A} is said to have the *small index property* if every subgroup of index less than 2^{\aleph_0} in $\text{Aut}(\mathbf{A})$ is open. Thus, the above mentioned result says that the countable structure over the empty signature has the small index property.

The small index property has strong consequences. Whenever \mathbf{A} and \mathbf{B} are countable structures, such that \mathbf{A} has the small index property, then every group-isomorphism from $\text{Aut}(\mathbf{A})$ to $\text{Aut}(\mathbf{B})$ is continuous. By a result by Lascar [23, Corollary 2.8], every continuous isomorphism between the automorphism groups of countable structures is already a homeomorphism. It follows that whenever \mathbf{A} and \mathbf{B} are countable structures and \mathbf{A} has the small index property then every group-isomorphism between $\text{Aut}(\mathbf{A})$ and $\text{Aut}(\mathbf{B})$ is a homeomorphism. Summing up, a countable structure with the small index property is determined among all other countable structures by its automorphism group (considered as abstract group) up to first order bi-interpretability.

This observation has been spurring the interest into structures with the small index property. A few corner-stones in the research about the small index property include [10, 16, 17, 18, 19, 21, 31, 33].

Another, rather different, approach to the reconstruction of ω -categorical structures is due to Rubin [30] — based on (weak) $\forall\exists$ -interpretations. It would go too far to describe this method at this place. However, if a countable ω -categorical structure \mathbf{A} has a weak $\forall\exists$ -interpretation and if \mathbf{B} is another ω -categorical structure, then every isomorphism between the automorphism groups of \mathbf{A} and \mathbf{B} is a homeomorphism (cf. also [2, 3]). Thus, a countable, ω -categorical structure with a weak $\forall\exists$ -interpretation is determined among the ω -categorical structures by its automorphism group (considered as an abstract group) up to first order bi-interpretability.

The automorphism groups of first order structures have been the topic of intensive research. Much less is known about their endomorphism monoids. This situation is slowly changing as is witnessed by the papers [6, 9, 11, 12, 22, 24, 27, 29], that deal with such diverse topics like the Bergman property, cofinality and strong cofinality, universality, idempotents, generic elements, and ideals of the endomorphism monoids of countable homogeneous structures.

In this paper we will study the question, how much information about a relational structure can be recovered from its endomorphism monoid (considered as an abstract monoid). In particular, inspired by the group case and by a recent paper by Bodirsky, Pinsker and Pongrácz [5], we study when the endomorphism monoid of a relational structure \mathbf{A}

has the property that every monoid isomorphism to the endomorphism monoid of a structure \mathbf{B} from a given class \mathcal{K} is automatically already a homeomorphism (here, the topology in question is always the canonical topology of pointwise convergence). Using the terminology of [5], this means, that $\text{End}(\mathbf{A})$ has *automatic homeomorphicity with respect to \mathcal{K}* .

In [5] automatic homeomorphicity was shown for

- the monoid of injective functions on \mathbb{N} ,
- the full transformation monoid on \mathbb{N} ,
- the monoid of homomorphic self-embeddings of the Rado graph, and
- the endomorphism monoid of the Rado graph,
- the monoid of homomorphic self-embeddings of the countable universal homogeneous digraph.

Moreover, automatic homeomorphicity with respect to the class of countable ω -categorical structures was shown for

- the endomorphism monoid of the countable universal homogeneous tournament,
- the monoid of homomorphic self-embeddings of the countable universal homogeneous k -uniform hypergraph.

Recently we learnt that Truss, Vergas-García [34] and Hyde [20] showed independently the automatic homeomorphicity for the endomorphism monoid of $(\mathbb{Q}, <)$.

We will extend this list by proving that the endomorphism monoid of the rationals (\mathbb{Q}, \leq) , the endomorphism monoid of the countable universal homogeneous poset (\mathbb{P}, \leq) , and the monoid of non-expansive self-maps of the rational Urysohn-space all have automatic homeomorphicity with respect to the class of countable structures whose endomorphism monoids have just finitely many weak orbits (in the sense of [32]), cf. Theorem 4.11, Theorem 4.15, and Theorem 4.13.

1. PRELIMINARIES

1.1. Transformation monoids. For a set A , the set of all function from A to A , equipped with composition of functions, forms a monoid \mathfrak{T}_A . The submonoids of \mathfrak{T}_A are called *transformation monoids on A* . The monoid \mathfrak{T}_A is also called the *full transformation monoid on A* .

The submonoid of \mathfrak{T}_A that consists of all permutations on A is called the *full symmetric group on A* . It will be denoted by \mathfrak{S}_A .

If we equip A with the discrete topology, then \mathfrak{T}_A is a product space of A . Thus, it is canonically equipped with the Tychonoff topology (a.k.a. the topology of pointwise convergence). For every $h \in \mathfrak{T}_A$, and for every finite subset M of A , we consider the set

$$\Phi_{h,M} := \{f \in \mathfrak{T}_A \mid f \upharpoonright_M = h \upharpoonright_M\}.$$

Then

$$\{\Phi_{h,M} \mid h \in \mathfrak{T}_A, M \subseteq A \text{ finite}\}.$$

forms a basis of the Tychonoff topology on \mathfrak{T}_A . Moreover, every transformation monoid on A is canonically equipped with the corresponding subspace topology.

If A is countably infinite, then the topology on \mathfrak{T}_A is metrizable by an ultrametric. In particular, if $\bar{a} = (a_i)_{i \in \omega}$ is an enumeration of A , then we consider the function

$$D_{\bar{a}}: \mathfrak{T}_A \times \mathfrak{T}_A \rightarrow \omega^+$$

$$(f, g) \mapsto \begin{cases} \min\{i \in \omega \mid f(a_i) \neq g(a_i)\} & f \neq g \\ \omega & f = g. \end{cases}$$

Finally, the mentioned ultrametric on \mathfrak{T}_A is given by

$$d_{\bar{a}}(f, g) := \begin{cases} 2^{-D_{\bar{a}}(f, g)} & f \neq g \\ 0 & f = g, \end{cases}$$

for all $f, g \in \mathfrak{T}_A$.

It is easy to see that for every enumeration \bar{a} of A and for all $f, g, h \in \mathfrak{T}_A$ we have $d_{\bar{a}}(f, g) \leq d_{\bar{a}}(h \circ f, h \circ g)$, and that equality holds if h is injective. In other words, the metric $d_{\bar{a}}$ is *left- \mathfrak{T}_A -subinvariant*.

In the following, whenever we deal with a transformation monoid $\mathfrak{M} \leq \mathfrak{T}_A$, we implicitly consider it to be equipped with the topology of pointwise convergence.

1.2. Relational structures. A *relational signature* is a pair $\underline{\Sigma} = (\Sigma, \text{ar})$ where Σ is a set of *relational symbols* and $\text{ar}: \Sigma \rightarrow \mathbb{N} \setminus \{0\}$ assigns to each relational symbol its *arity*. With $\Sigma^{(n)}$ we will denote the set of all n -ary relational symbols in Σ .

A $\underline{\Sigma}$ -*structure* \mathbf{A} is a pair $(A, (\varrho^{\mathbf{A}})_{\varrho \in \Sigma})$, such that A is a set, and such that for each $\varrho \in \Sigma$ we have that $\varrho^{\mathbf{A}}$ is a relation of arity $\text{ar}(\varrho)$ on A . The set A will be called the *carrier* of \mathbf{A} and the relations $\varrho^{\mathbf{A}}$ will be called the *basic relations* of \mathbf{A} . If the signature $\underline{\Sigma}$ is of no importance, we will speak only about relational structures. If not said otherwise the carrier of a $\underline{\Sigma}$ -structure \mathbf{A} will always be denoted by A and the basic relations of \mathbf{A} will be denoted by $\varrho^{\mathbf{A}}$ for each $\varrho \in \Sigma$.

Let \mathbf{A} and \mathbf{B} be $\underline{\Sigma}$ -structures. A function $h: A \rightarrow B$ is called a *homomorphism* if for all $n \in \mathbb{N} \setminus \{0\}$, for all $\varrho \in \Sigma^{(n)}$ and for all $\bar{a} = (a_1, \dots, a_n) \in \varrho^{\mathbf{A}}$ we have that $h(\bar{a}) := (h(a_1), \dots, h(a_n)) \in \varrho^{\mathbf{B}}$. A function $h: A \rightarrow B$ is called *embedding* if h is injective and if for all $n \in \mathbb{N} \setminus \{0\}$, for all $\varrho \in \Sigma^{(n)}$ and for all $\bar{a} \in A^n$ we have

$$\bar{a} \in \varrho^{\mathbf{A}} \iff h(\bar{a}) \in \varrho^{\mathbf{B}}.$$

Surjective embeddings are called *isomorphisms*. As usual, isomorphisms of a relational structure \mathbf{A} onto itself are called *automorphisms*, and homomorphisms of \mathbf{A} to itself are called *endomorphisms*. Moreover,

embeddings of \mathbf{A} into itself will be called *selfembeddings* of \mathbf{A} . The set of all automorphisms, endomorphisms, and selfembeddings of \mathbf{A} will be denoted by $\text{Aut}(\mathbf{A})$, $\text{End}(\mathbf{A})$, and $\text{Emb}(\mathbf{A})$, respectively. Clearly, $\text{Aut}(\mathbf{A})$ is a permutation group, and $\text{End}(\mathbf{A})$ and $\text{Emb}(\mathbf{A})$ are transformation monoids.

Another word about notation: Whenever we write $h: \mathbf{A} \rightarrow \mathbf{B}$, we mean that h is a homomorphism from \mathbf{A} to \mathbf{B} . Moreover, with $h: \mathbf{A} \hookrightarrow \mathbf{B}$ we denote the fact that h is an embedding from \mathbf{A} into \mathbf{B} .

Example 1.1. Consider the relational signature $\underline{\Sigma}_M$ that contains for every $r \in \mathbb{Q}^+ \cup \{0\}$ a binary relational symbol ϱ_r . Then to every metric space (A, d) we may associate a Σ_M -structures \mathbf{A} by defining

$$\varrho_r^{\mathbf{A}} := \{(x, y) \in A^2 \mid d(x, y) \leq r\},$$

for every $r \in \mathbb{Q}^+ \cup \{0\}$. The metric d can be reconstructed from \mathbf{A} by

$$d(x, y) = \inf\{r \in \mathbb{Q}^+ \cup \{0\} \mid (x, y) \in \varrho_r^{\mathbf{A}}\}.$$

To make this correspondence functorial, the proper choice of morphisms between metric spaces are the non-expansive maps. Recall that a function $f: (A, d_A) \rightarrow (B, d_B)$ is called *non-expansive* if for all $x, y \in A$ we have

$$d_B(f(x), f(y)) \leq d_A(x, y).$$

With this definition of morphisms between metric spaces, the assignment $R: (A, d) \mapsto \mathbf{A}$, $R: f \mapsto f$ is a full embedding into the category $\mathfrak{C}_{\underline{\Sigma}_M}$ of all $\underline{\Sigma}_M$ -structures with homomorphisms as morphisms. Therefore, in the following we will identify metric spaces with their relational counter-parts.

1.3. Homogeneous relational structures. Following Fraïssé, for every $\underline{\Sigma}$ -structure \mathbf{A} , its age is the class of all finite $\underline{\Sigma}$ -structures that are embeddable into \mathbf{A} . It will be denoted by $\text{Age}(\mathbf{A})$. A $\underline{\Sigma}$ -structure \mathbf{B} is called *younger* than \mathbf{A} if $\text{Age}(\mathbf{B}) \subseteq \text{Age}(\mathbf{A})$. By $\overline{\text{Age}(\mathbf{A})}$ we will denote the class of all countable $\underline{\Sigma}$ -structures younger than \mathbf{A} .

Definition 1.2. A countable $\underline{\Sigma}$ -structure \mathbf{A} is called *universal* if every structure from $\overline{\text{Age}(\mathbf{A})}$ can be embedded into \mathbf{A} . It is called *homogeneous* if for every $\mathbf{B} \in \text{Age}(\mathbf{A})$ and for all embeddings $\iota_1, \iota_2: \mathbf{B} \hookrightarrow \mathbf{A}$ there exists an automorphism h of \mathbf{A} such that $\iota_2 = h \circ \iota_1$.

Remark. Our definition of homogeneity is equivalent to the more usual definition that every isomorphism between finite substructures of \mathbf{A} extends to an automorphism. Indeed, ι_1 and ι_2 mark two isomorphic copies of \mathbf{B} in \mathbf{A} , and at the same time define an isomorphism between these two finite substructures given by $g: \iota_1(\mathbf{B}) \rightarrow \iota_2(\mathbf{B}): x \mapsto \iota_2(\iota_1^{-1}(x))$. Finally the postulated automorphism h extends g . On the other hand, every isomorphism g between finite substructures \mathbf{B}_1 and \mathbf{B}_2 of \mathbf{A} defines two embeddings $\iota_1: \mathbf{B}_1 \hookrightarrow \mathbf{A}$ and $\iota_2: \mathbf{B}_2 \hookrightarrow \mathbf{A}$, where

ι_1 is the identical embedding and $\iota_2 = g \circ \iota_1$. Then every automorphism h of \mathbf{A} that extends g will satisfy $\iota_2 = h \circ \iota_1$.

Definition 1.3. Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures. We say that \mathcal{C} has the

hereditary property (HP): if whenever $\mathbf{A} \in \mathcal{C}$ and \mathbf{B} is a $\underline{\Sigma}$ -structure embeddable into \mathbf{A} , then also $\mathbf{B} \in \mathcal{C}$,

joint embedding property (JEP): if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ there exists a $\mathbf{C} \in \mathcal{C}$ and embeddings $f: \mathbf{A} \hookrightarrow \mathbf{C}$ and $g: \mathbf{B} \hookrightarrow \mathbf{C}$,

amalgamation property (AP): if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ from \mathcal{C} and for all embeddings $f: \mathbf{A} \hookrightarrow \mathbf{B}$, $g: \mathbf{A} \hookrightarrow \mathbf{C}$, there exists $\mathbf{D} \in \mathcal{C}$ and embeddings $\hat{f}: \mathbf{C} \hookrightarrow \mathbf{D}$, $\hat{g}: \mathbf{B} \hookrightarrow \mathbf{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\hat{f}} & \mathbf{D} \\ \uparrow g & & \uparrow \hat{g} \\ \mathbf{A} & \xrightarrow{f} & \mathbf{B} \end{array}$$

Let us recall the well-known characterization of ages of countable structures and, in particular, of countable homogeneous structures, by Roland Fraïssé:

Theorem 1.4 (Fraïssé ([13])). *Let \mathcal{C} be a class of finite $\underline{\Sigma}$ -structures. Then \mathcal{C} is equal to the age of a countable structure if and only if it has up to isomorphism just countably many members, and it has the HP and the JEP. Moreover, \mathcal{C} is equal to the age of a countable homogeneous structure if and only if it has in addition the AP. Finally, any two countable homogeneous $\underline{\Sigma}$ -structures with the same age are isomorphic.*

Definition 1.5. A class of finite $\underline{\Sigma}$ -structures is called an *age* if it has the HP, the JEP, and if it contains up to isomorphism just countably many structures. An age is called a *Fraïssé-class* if it has the AP. A countable homogeneous $\underline{\Sigma}$ -structure \mathbf{U} is called the *Fraïssé-limit* of $\text{Age}(\mathbf{U})$.

Example 1.6. Some examples of Fraïssé-classes include:

- the class of finite simple graphs,
- the class of finite posets (strictly or non-strictly ordered),
- the class of finite linear orders (strictly or non-strictly ordered)
- the class of finite metric spaces with rational distances,
- the class of finite metric spaces with rational distances ≤ 1 ,
- the class of finite tournaments.

The corresponding Fraïssé-limits are the Rado graph (aka. the countable random graph), the countable generic poset, the rationals, the rational Urysohn space, the rational Urysohn sphere, and the random tournament, respectively.

1.4. Homomorphism-homogeneous relational structures. In [8], Cameron and Nešetřil introduced several variants of the notion of homogeneity. One of these variations is homomorphism-homogeneity:

Definition 1.7. A countable $\underline{\Sigma}$ -structure \mathbf{U} is called *homomorphism-homogeneous* if for every $\mathbf{A} \in \text{Age}(\mathbf{U})$, for all embeddings $\iota: \mathbf{A} \hookrightarrow \mathbf{U}$, and for all homomorphisms $h: \mathbf{A} \rightarrow \mathbf{U}$ there exists an endomorphism \hat{h} of \mathbf{U} such that $h = \hat{h} \circ \iota$.

Remark. This definition of homomorphism homogeneity slightly differs from the original given definition in [8]. However, the equivalence of our definition to the original one is obvious.

The connection between the notions of homogeneity and homomorphism-homogeneity was created by Dolinka (cf. [12, Proposition 3.8]):

Definition 1.8. Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures. We say that \mathcal{C} has the *homo-amalgamation property* (HAP) if for all $\mathbf{A}, \mathbf{B} \in \mathcal{C}$, $g: \mathbf{A} \hookrightarrow \mathbf{B}$, $\mathbf{T}_1 \in \mathcal{C}$, $a: \mathbf{A} \rightarrow \mathbf{T}_1$ there exist $\mathbf{T}_2 \in \mathcal{C}$, $b: \mathbf{B} \rightarrow \mathbf{T}_2$, $h: \mathbf{T}_1 \hookrightarrow \mathbf{T}_2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{\quad b \quad} & \mathbf{T}_2 \\ \uparrow g & & \uparrow h \\ \mathbf{A} & \xrightarrow{\quad a \quad} & \mathbf{T}_1 \end{array}$$

Proposition 1.9 ([12, Proposition 3.8]). *Let \mathbf{U} be a countable homogeneous structures. Then \mathbf{U} is homomorphism-homogeneous if and only if its age has the HAP.*

Example 1.10. Given the rather extensive literature on the classification of homomorphism-homogeneous structures, Proposition 1.9 is a convenient tool for showing that the age of a given homogeneous structure has the HAP:

- By [8, Proposition 2.1] the Rado graph is homomorphism-homogeneous. Thus, the class of finite graphs has the HAP.
- By [7, Proposition 25] and [28, Theorem 4.5], the countable generic poset (\mathbb{P}, \leq) is homomorphism-homogeneous. Thus, the class of finite posets has the HAP.
- By [7, Proposition 15]) the countable generic strict poset $(\mathbb{P}, <)$ is homomorphism-homogeneous. Thus, the class of finite strict partial orders has the HAP.
- By [7, Proposition 25] and [28, Theorem 4.5], we have that the structure (\mathbb{Q}, \leq) is homomorphism-homogeneous. Thus, the class of finite linear orders has the HAP.
- By [7, Proposition 15] the structure $(\mathbb{Q}, <)$ is homomorphism-homogeneous. Thus, the class of finite strict linear orders has the HAP.

For another group of ages we observe the HAP in a more direct way:

- It was shown in [12, Lemma 3.5] that the class of finite metric spaces with rational distances has the HAP.
- The same construction as in [12, Lemma 3.5] shows that the class of finite metric spaces with rational distances ≤ 1 has the HAP.
- Every homomorphism between tournaments is an embedding. Thus, since the class of finite tournaments has the AP, it follows that it also has the HAP.

On the other hand, a number of prominent countable homogeneous structures fails to be homomorphism-homogeneous, and thus, their ages do not have the HAP. This list includes the Henson graphs (cf. [14]) and the Henson digraphs (cf. [15]).

2. UNIVERSAL HOMOGENEOUS ENDOMORPHISMS

Definition 2.1. Let \mathbf{U}, \mathbf{A} be relational structures of the same type, let u be an endomorphism of \mathbf{U} , and let $h: \mathbf{A} \rightarrow \mathbf{U}$ be a homomorphism. If there exists an embedding $\iota: \mathbf{A} \hookrightarrow \mathbf{U}$ such that $h = u \circ \iota$, then we say that h *factors through u by ι* .

Definition 2.2. Let \mathbf{U} be a countable relational structure. An endomorphism u of \mathbf{U} is called *universal* if for every $\mathbf{A} \in \text{Age}(\mathbf{U})$ we have that every homomorphism $h: \mathbf{A} \rightarrow \mathbf{U}$ factors through u by some embeddings $\iota: \mathbf{A} \hookrightarrow \mathbf{U}$.

Definition 2.3. Let \mathbf{U} be a countable relational structure. An endomorphism u of \mathbf{U} is called *homogeneous* if for every $\mathbf{A} \in \text{Age}(\mathbf{U})$, for every homomorphism $h: \mathbf{A} \rightarrow \mathbf{U}$, and for all factorization $h = u \circ \iota_1 = u \circ \iota_2$ by embeddings $\iota_1, \iota_2: \mathbf{A} \hookrightarrow \mathbf{U}$, there exists an automorphism f of \mathbf{U} , such that $f \circ \iota_1 = \iota_2$, and such that $u \circ f = u$.

Remark. Universal homogeneous endomorphisms were introduced in [29], where they were mainly used for the characterization of retracts of homogeneous structures.

Definition 2.4. Let \mathcal{C} be a class of $\underline{\Sigma}$ -structures. We say that \mathcal{C} has the *amalgamated extension property* (AEP) if for all $\mathbf{A}, \mathbf{B}_i, \mathbf{T} \in \mathcal{C}$, $f_i: \mathbf{A} \hookrightarrow \mathbf{B}_i$, $h_i: \mathbf{B}_i \rightarrow \mathbf{T}$ (where $i \in \{1, 2\}$), with $h_1 \circ f_1 = h_2 \circ f_2$, there exist $\mathbf{C} \in \mathcal{C}$, $g_i: \mathbf{B}_i \hookrightarrow \mathbf{C}$ (where $i \in \{1, 2\}$), $\mathbf{T}' \in \mathcal{C}$, $h: \mathbf{C} \rightarrow \mathbf{T}'$,

$k: \mathbf{T} \hookrightarrow \mathbf{T}'$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & & & \mathbf{T}' \\
 & & & \nearrow h & \\
 & & \mathbf{T} & \nearrow k & \\
 & \nearrow h_1 & & & \\
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} & & \\
 \uparrow f_1 & & \uparrow g_2 & & \\
 \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2 & \nearrow h_2 &
 \end{array}$$

The following is a complete characterization of all countable homogeneous structures that have a universal homogeneous endomorphism:

Proposition 2.5 ([29, Proposition 4.7]). *Let \mathbf{U} be a countably infinite homogeneous structure. Then \mathbf{U} has a universal homogeneous endomorphism if and only if $\text{Age}(\mathbf{U})$ has the AEP and the HAP.*

Definition 2.6. For a class \mathcal{C} of $\underline{\Sigma}$ -structures, by $(\mathcal{C}, \rightarrow)$ we will denote the category that has the elements of \mathcal{C} as objects and all homomorphisms between the elements of \mathcal{C} as morphisms. Analogously, by $(\mathcal{C}, \hookrightarrow)$ we will denote the subcategory of $(\mathcal{C}, \rightarrow)$ whose morphisms are all embeddings between structures of \mathcal{C} .

The following is going to be useful in order to identify relational structures whose age has the AEP:

Definition 2.7 ([11, Section 1.1]). Let \mathbf{U} be a countably infinite $\underline{\Sigma}$ -structure. Then we say that $\text{Age}(\mathbf{U})$ has the *strict amalgamation property* (strict AP) if for all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \text{Age}(\mathbf{U})$, and for all embeddings $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$ there exists some $\mathbf{C} \in \text{Age}(\mathbf{U})$ and embeddings $g_1: \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \hookrightarrow \mathbf{C}$ such that the following is a *pushout-square* in the category $(\overline{\text{Age}(\mathbf{U})}, \rightarrow)$:

$$\begin{array}{ccc}
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\
 \uparrow f_1 & & \uparrow g_2 \\
 \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2.
 \end{array}$$

That is, if $\mathbf{T} \in \overline{\text{Age}(\mathbf{U})}$, and if $h_1: \mathbf{B}_1 \rightarrow \mathbf{T}$, $h_2: \mathbf{B}_2 \rightarrow \mathbf{T}$ are homomorphism such that $h_1 \circ f_1 = h_2 \circ f_2$, then there exists a unique

homomorphism $h: \mathbf{C} \rightarrow \mathbf{T}$, such that the following diagram commutes:

$$\begin{array}{ccc}
 & & \mathbf{T} \\
 & \nearrow h_1 & \\
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\
 \uparrow f_1 & & \uparrow g_2 \\
 \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow h \\
 \searrow h_2
 \end{array}$$

Lemma 2.8. *Let \mathbf{U} be a countable $\underline{\Sigma}$ -structure whose age has the strict AP. Then $\text{Age}(\mathbf{U})$ has the AEP, too.*

Proof. Let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2, \mathbf{T} \in \text{Age}(\mathbf{U})$, let $f_1: \mathbf{A} \hookrightarrow \mathbf{B}_1$, $f_2: \mathbf{A} \hookrightarrow \mathbf{B}_2$ be embeddings and let $h_1: \mathbf{B}_1 \rightarrow \mathbf{T}$, $h_2: \mathbf{B}_2 \rightarrow \mathbf{T}$, such that $h_1 \circ f_1 = h_2 \circ f_2$. Since $\text{Age}(\mathbf{U})$ has the strict amalgamation property, there exists a $\mathbf{C} \in \text{Age}(\mathbf{U})$, and embeddings $g_1: \mathbf{B}_1 \hookrightarrow \mathbf{C}$, $g_2: \mathbf{B}_2 \hookrightarrow \mathbf{C}$, such that the following diagram is a pushout-square in $(\text{Age}(\mathbf{U}), \rightarrow)$:

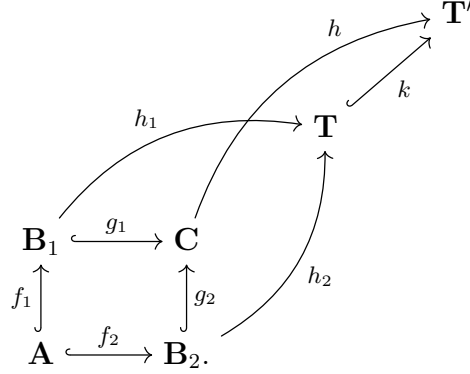
$$(1) \quad
 \begin{array}{ccc}
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\
 \uparrow f_1 & & \uparrow g_2 \\
 \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2
 \end{array}$$

Since $h_1 \circ f_1 = h_2 \circ f_2$, and since (1) is a pushout-square, there exists a unique homomorphism $h: \mathbf{C} \rightarrow \mathbf{T}$ that makes the following diagram commutative:

$$\begin{array}{ccc}
 & & \mathbf{T} \\
 & \nearrow h_1 & \\
 \mathbf{B}_1 & \xrightarrow{g_1} & \mathbf{C} \\
 \uparrow f_1 & & \uparrow g_2 \\
 \mathbf{A} & \xrightarrow{f_2} & \mathbf{B}_2
 \end{array}
 \quad
 \begin{array}{c}
 \nearrow h \\
 \searrow h_2
 \end{array}$$

Now, we can put $\mathbf{T}' := \mathbf{T}$, and we can define $k: \mathbf{T} \hookrightarrow \mathbf{T}'$ to be the identical embedding, and we obtain that the following diagram commutes,

too:



This shows, that $\text{Age}(\mathbf{U})$ has the AEP. \square

A special case of the strict amalgamation property is the free amalgamation property:

Definition 2.9. Let \mathbf{A} , \mathbf{B}_1 , \mathbf{B}_2 be $\underline{\Sigma}$ -structures, such that $\mathbf{A} \leq \mathbf{B}_1$, $\mathbf{A} \leq \mathbf{B}_2$, and such that $B_1 \cap B_2 = A$. Then the *amalgamated free sum* of \mathbf{B}_1 and \mathbf{B}_2 with respect to \mathbf{A} is the $\underline{\Sigma}$ -structure $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$ with carrier $B_1 \cup B_2$, such that for each $\varrho \in \Sigma$ we have

$$\varrho^{\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2} = \varrho^{\mathbf{B}_1} \cup \varrho^{\mathbf{B}_2}$$

Definition 2.10 (cf. [26, Page 1602]). An age \mathcal{C} of $\underline{\Sigma}$ -structures is said to have the *free amalgamation property* (free AP) if \mathcal{C} is closed with respect to amalgamated free sums. A homogeneous structures whose age has the free amalgamation property is called *free homogeneous*.

Lemma 2.11. Let \mathbf{U} be a countably infinite $\underline{\Sigma}$ -structure, such that $\text{Age}(\mathbf{U})$ has the free amalgamation property. Then $\text{Age}(\mathbf{U})$ has the strict amalgamation property, too.

Proof. It is easy to see that the if \mathbf{A} , \mathbf{B}_1 , and \mathbf{B}_2 are $\underline{\Sigma}$ -structures with $\mathbf{A} \leq \mathbf{B}_1$, $\mathbf{A} \leq \mathbf{B}_2$, and $B_1 \cap B_2 = A$, then the following is a pushout-square in the category of all $\underline{\Sigma}$ -structures:

$$\begin{array}{ccc} \mathbf{B}_1 & \xrightarrow{=} & \mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2 \\ \uparrow & & \uparrow \\ \mathbf{A} & \xrightarrow{=} & \mathbf{B}_2. \end{array}$$

Since $(\overline{\text{Age}(\mathbf{U})}, \rightarrow)$ is a full subcategory of the category of all $\underline{\Sigma}$ -structures, it follows that amalgamated free sums in $\text{Age}(\mathbf{U})$ are pushouts in $(\overline{\text{Age}(\mathbf{U})}, \rightarrow)$, too. Consequently, $\text{Age}(\mathbf{U})$ has the strict amalgamation property. \square

Example 2.12. Often it is easier to observe the strict AP rather than the AEP. In particular, the ages of the following relational structures have the strict AP, and have therefore also the AEP:

- the Rado graph (because it is a free homogeneous structure),
- the countable generic poset (\mathbb{P}, \leq) (cf. [12, Pages 7,8]),
- the countable generic strict poset $(\mathbb{P}, <)$ (by the same argument as for (\mathbb{P}, \leq)),
- the Henson-graphs (because they are free homogeneous structures, cf. [26, Example 2.2.2]),
- the Henson-digraphs (because they are free homogeneous structures, cf. [26, Page 1604]).

There is also a number of ages with the AEP but without the strict AP:

- The class of finite tournaments has the AP. Since every homomorphism between tournaments is an embedding, it follows that the class of all finite tournaments trivially fulfills the AEP.
- For the same reason as above, the class of finite strict linear orders satisfies the AEP.
- The class of finite (non-strict) linear orders satisfies the AEP (cf. [22, Proposition 3.23]).
- The class of finite metric spaces with rational distances has the AEP (implicit in [22]).
- The class of finite metric spaces with rational distances ≤ 1 has the AEP (implicit in [22]).

Using Example 1.10 together with Proposition 2.5, we obtain that the following structures have universal homogeneous endomorphisms:

- the Rado graph,
- the countable generic tournament,
- the countable generic strict poset $(\mathbb{P}, <)$,
- the countable generic poset (\mathbb{P}, \leq) ,
- the rationals with strict order $(\mathbb{Q}, <)$,
- the rationals with the non-strict order (\mathbb{Q}, \leq) ,
- the rational Urysohn-space,
- the rational Urysohn-sphere.

Remark. For some structures we can give an explicit description of a universal homogeneous endomorphism. For the countable generic tournament and for $(\mathbb{Q}, <)$ the identical automorphism is a universal homogeneous endomorphism. For (\mathbb{Q}, \leq) a universal homogeneous endomorphism was described in [29, Remark on page 32]. In [24], a generic endomorphism of (\mathbb{Q}, \leq) was described. This endomorphism turns out to be universal homogeneous in our sense. It would be interesting to examine the relations between generic endomorphisms and universal homogeneous endomorphism.

3. STRONG GATE COVERINGS

Definition 3.1. Let A be a countably infinite set, let $\mathfrak{M} \leq \mathfrak{T}_A$ be a transformation monoid, let \mathfrak{G} be the group of units in \mathfrak{M} , and let $\overline{\mathfrak{G}}$ be

the closure of \mathfrak{G} in \mathfrak{M} . Then we say that \mathfrak{M} has a *strong gate covering* if there exists an open covering \mathcal{U} of \mathfrak{M} and elements $f_U \in U$, for every $U \in \mathcal{U}$, such that for all $U \in \mathcal{U}$ and for all Cauchy-sequences $(g_n)_{n \in \mathbb{N}}$ of elements from U there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n)_{n \in \mathbb{N}}$ of elements from $\overline{\mathfrak{G}}$ such that for all $n \in \mathbb{N}$ we have

$$g_n = \kappa_n \circ f_U \circ \iota_n.$$

Remark. Strong gate coverings appear implicitly for the first time in [5]. In particular, it is shown there that the endomorphism monoid of the Rado graph has a strong gate covering.

Lemma 3.2. *Let \mathbf{U} be a relational structure that has a universal homogeneous endomorphism u . Let \mathbf{A} be a finite substructure of \mathbf{U} . Let f, g be endomorphisms of \mathbf{U} that agree on A . Then there exist self-embeddings ι_1, ι_2 , such that*

- (1) $f = u \circ \iota_1$,
- (2) $g = u \circ \iota_2$,
- (3) $\iota_1 \upharpoonright_A = \iota_2 \upharpoonright_A$.

Proof. Since u is universal, there exist selfembeddings ι_1 and ι_2 of \mathbf{U} , such that

$$\begin{aligned} f &= u \circ \iota_1, \\ g &= u \circ \iota_2. \end{aligned}$$

Let $\hat{\iota}_i := \iota_i \upharpoonright_A$, for $i \in \{1, 2\}$, and let $\hat{f} := f \upharpoonright_A$. Let $a \in A$. Then we compute

$$\hat{f}(a) = f(a) = u(\iota_1(a)) = u(\hat{\iota}_1(a)).$$

Moreover,

$$\hat{f}(a) = f(a) = g(a) = u(\iota_2(a)) = u(\hat{\iota}_2(a)).$$

Since u is homogeneous, there exists an automorphism h of \mathbf{U} , such that $h \circ \hat{\iota}_1 = \hat{\iota}_2$, and such that $u \circ h = u$. Let $\tilde{\iota}_1 := h \circ \iota_1$. Then $\tilde{\iota}_1 \upharpoonright_A = h \circ \hat{\iota}_1 = \hat{\iota}_2 = \iota_2 \upharpoonright_A$. Moreover, we have

$$u \circ \tilde{\iota}_1 = u \circ h \circ \iota_1 = u \circ \iota_1 = f.$$

□

Proposition 3.3. *Let \mathbf{U} be a countably infinite relational structure that has a universal homogeneous endomorphism u . Let $(f_j)_{j < \omega}$ be a sequence of endomorphisms of \mathbf{U} that converge to an endomorphism f of \mathbf{U} . Then there is a sequence $(\iota_j)_{j < \omega}$ of homomorphic selfembeddings of \mathbf{U} , such that*

- (1) for every $j < \omega$ we have $f_j = u \circ \iota_j$,
- (2) $(\iota_j)_{j < \omega}$ converges to $\iota \in \text{Emb}(\mathbf{U})$,
- (3) $f = u \circ \iota$.

Proof. Since u is a universal homogeneous endomorphism of \mathbf{U} , there exists a selfembedding ι of \mathbf{U} such that $f = u \circ \iota$.

Let $\bar{a} = (a_i)_{i \in \omega}$ be any enumeration of U . For every finite substructure \mathbf{A} of \mathbf{U} let $n_{\mathbf{A}}$ be the smallest element of ω such that $A \subseteq \{a_0, \dots, a_{n_{\mathbf{A}}-1}\}$.

Let $(\mathbf{A}_i)_{i < \omega}$ be a sequence of finite substructures of \mathbf{U} such that $\mathbf{A}_i \leq \mathbf{A}_j$ whenever $i \leq j$ and such that $\bigcup_i \mathbf{A}_i = \mathbf{U}$ (this exists because \mathbf{U} is countably infinite). Then the sequence $(n_{\mathbf{A}_i})_{i < \omega}$ is monotonous and unbounded.

Since $(f_j)_{j < \omega}$ converges to f , we have that for every $i < \omega$ there exists a $j_i < \omega$ such that for every $k > j_i$ we have that $D_{\bar{a}}(f_k, f) > n_{\mathbf{A}_i}$. Without loss of generality we may assume that j_i is chosen as small as possible.

For $0 \leq k < j_0$, using the fact that u is universal homogeneous, we choose ι_k , such that

$$f_k = u \circ \iota_k.$$

For $j_i \leq k < j_{i+1}$, using Lemma 3.2, we chose ι_k , such that

$$f_k = u \circ \iota_k,$$

and such that ι_k agrees with ι on A_i .

It remains to observe that the sequence $(\iota_j)_{j < \omega}$ converges to ι . Let $\varepsilon > 0$ and let $N := \max(-\lfloor \log_2(\varepsilon) \rfloor, 1)$. Then there exists an $i < \omega$, such that $\{a_0, \dots, a_{N-1}\} \subseteq A_i$. But then, by construction, for all $k \geq j_i$, we have that ι_k agrees with ι on $\{a_0, \dots, a_{N-1}\}$ — in particular, $D_{\bar{a}}(\iota_k, \iota) \geq N$, and thus $d_{\bar{a}}(\iota_k, \iota) \leq \varepsilon$. \square

Proposition 3.4. *If \mathbf{U} is a countable relational structure that has a universal homogeneous endomorphism, then $\text{End}(\mathbf{U})$ has a strong gate covering.*

Proof. This is a direct consequence of Propositions 3.3, taking $\mathcal{U} = \{\text{End}(\mathbf{U})\}$ as an open covering of $\text{End}(\mathbf{U})$, and using that $(\text{End}(\mathbf{U}), d_{\bar{a}})$ is a complete metric space, for each enumeration \bar{a} of U . \square

4. AUTOMATIC HOMEOMORPHICITY

Definition 4.1. Let \mathcal{K} be a class of structures and let $\mathbf{A} \in \mathcal{K}$. We say that $\text{End}(\mathbf{A})$ has *automatic homeomorphicity with respect to \mathcal{K}* if every monoid isomorphism from $\text{End}(\mathbf{A})$ to the endomorphism monoid of a member of \mathcal{K} is a homeomorphism.

Lemma 4.2. *Let A, B be countable sets, and let $\mathfrak{M}_1 \leq \mathfrak{T}_A$, $\mathfrak{M}_2 \leq \mathfrak{T}_B$ be monoids, such that \mathfrak{M}_1 has a dense set of units. Let \bar{a} and \bar{b} be enumerations of A and B , respectively.*

Let $h: \mathfrak{M}_1 \rightarrow \mathfrak{M}_2$ be a continuous homomorphism. Then h is uniformly continuous from $(\mathfrak{M}_1, d_{\bar{a}})$ to $(\mathfrak{M}_2, d_{\bar{b}})$.

Proof. Let e_1, e_2 be the neutral elements of \mathfrak{M}_1 and of \mathfrak{M}_2 , respectively. Let $\varepsilon > 0$. Since h is continuous at e_1 , there exists a $\Delta \in \mathbb{N} \setminus \{0\}$ such that, with $\delta := 2^{-\Delta}$, for all $m \in \mathfrak{M}_1$ with $d_{\bar{a}}(m, e_1) \leq \delta$ we have $d_{\bar{b}}(h(m), e_2) \leq \varepsilon$.

Let $m, m' \in \mathfrak{M}_1$ with $d_{\bar{a}}(m, m') \leq \delta$. Then we have

$$(m(a_0), \dots, m(a_{\Delta-1})) = (m'(a_0), \dots, m'(a_{\Delta-1})) =: \bar{c}.$$

But since the units lie dense in \mathfrak{M}_1 , there exists a unit $g \in \mathfrak{M}_1$ with

$$(g(a_0), \dots, g(a_{\Delta-1})) = \bar{c}.$$

Consider now $\tilde{m} := g^{-1}m$ and $\tilde{m}' := g^{-1}m'$. Then $d_{\bar{a}}(\tilde{m}, e_1) \leq \delta$ and $d_{\bar{a}}(\tilde{m}', e_1) \leq \delta$.

Now we compute

$$\begin{aligned} \varepsilon &\geq d_{\bar{b}}(h(\tilde{m}), e_2) = d_{\bar{b}}(h(g^{-1}m), e_2) = d_{\bar{b}}(h(g)^{-1}h(m), e_2) \\ &= d_{\bar{b}}(h(m), h(g)) \end{aligned}$$

In the same way we obtain $d_{\bar{b}}(h(m'), h(g)) \leq \varepsilon$. Hence, since $d_{\bar{b}}$ is an ultrametric, we have $d_{\bar{b}}(h(m), h(m')) \leq \varepsilon$. \square

We will need the following basic facts about metric spaces and uniformly continuous functions:

Lemma 4.3. *Let (\mathbb{M}_1, d_1) be a metric space and let (\mathbb{M}_2, d_2) be a complete metric space. Then every uniformly continuous function f from (\mathbb{M}_1, d_1) to (\mathbb{M}_2, d_2) has a unique uniformly continuous extension to the completion of (\mathbb{M}_1, d_1) .*

Lemma 4.4. *Let Met be the category of metric spaces with uniformly continuous functions. Let cMet be the full subcategory of Met spanned by all complete metric spaces. Let $U: \text{cMet} \hookrightarrow \text{Met}$ be the inclusion functor. Then U has a left-adjoint functor C , mapping each metric space \mathbb{M} to its completion $\overline{\mathbb{M}}$ and every uniformly continuous function $f: \mathbb{M}_1 \rightarrow \mathbb{M}_2$ to its unique extension $\hat{f}: \overline{\mathbb{M}}_1 \rightarrow \overline{\mathbb{M}}_2$.*

Proof. Folklore, cf. [25, Page 92] \square

Lemma 4.5. *Let A be a countably infinite set and let \mathfrak{G} be a closed subgroup of \mathfrak{S}_A . Let $\bar{a} = (a_i)_{i < \omega}$ be an enumeration of A . Then the closure of \mathfrak{G} in \mathfrak{T}_A coincides with the Cauchy-completion of \mathfrak{G} in $(\mathfrak{T}_A, d_{\bar{a}})$.*

Proof. This follows immediately from the fact that $(\mathfrak{T}_A, d_{\bar{a}})$ is a complete metric space, and that complete subspaces of complete metric spaces are closed, and, vice versa, closed subspaces of complete metric spaces are complete. \square

Proposition 4.6. *Let \mathbf{A} and \mathbf{B} be two countable relational structures, such that $\text{End}(\mathbf{B})$ has a strong gate covering. Let $h: \text{End}(\mathbf{A}) \rightarrow \text{End}(\mathbf{B})$ be a continuous monoid-isomorphism. Then h is a homeomorphism.*

Before coming to the proof, let us recall the similar result by Lascar for closed permutation groups:

Proposition 4.7 ([23, Corollary 2.8]). *Let \mathbf{A} and \mathbf{B} be countable structures and let $f : \text{Aut}(\mathbf{A}) \rightarrow \text{Aut}(\mathbf{B})$ be a continuous isomorphism. Then f is a homeomorphism.*

Proof of Proposition 4.6. Let \bar{a} and \bar{b} be enumerations of A and B , respectively. Let $d_{\bar{a}}$ and $d_{\bar{b}}$ be the ultrametrics induced by \bar{a} and \bar{b} in \mathfrak{T}_A and \mathfrak{T}_B , respectively. Let further $f := h|_{\text{Aut}(\mathbf{A})}$.

Since h is continuous, we have that $f : \text{Aut}(\mathbf{A}) \rightarrow \text{Aut}(\mathbf{B})$ is continuous, too. By Proposition 4.7, f is a homeomorphism. Thus, by Lemma 4.2, $f : (\text{Aut}(\mathbf{A}), d_{\bar{a}}) \rightarrow (\text{Aut}(\mathbf{B}), d_{\bar{b}})$ and $f^{-1} : (\text{Aut}(\mathbf{B}), d_{\bar{b}}) \rightarrow (\text{Aut}(\mathbf{A}), d_{\bar{a}})$ are uniformly continuous, i.e., they are isomorphisms in the category Met .

Let $\hat{f} := C(f)$ be the unique extension of f to $\overline{\text{Aut}(\mathbf{A})}$ as a uniformly continuous function. Then \hat{f} is an isomorphism in the category cMet and $C(f^{-1}) = C(f)^{-1}$.

Let now $g := h|_{\overline{\text{Aut}(\mathbf{A})}}$. Since h is continuous, we have that $g : \overline{\text{Aut}(\mathbf{A})} \rightarrow \overline{\text{Aut}(\mathbf{B})}$ is continuous, too. Thus, from Lemma 4.2 we conclude that $g : (\overline{\text{Aut}(\mathbf{A})}, d_A) \rightarrow (\overline{\text{Aut}(\mathbf{B})}, d_B)$ is uniformly continuous. Since, clearly, we have $g|_{\text{Aut}(\mathbf{A})} = f$, we conclude from Lemma 4.3, that $g = C(f) = \hat{f}$. Thus $g : \overline{\text{Aut}(\mathbf{A})} \rightarrow \overline{\text{Aut}(\mathbf{B})}$ is a homeomorphism.

Now we are ready to show that h^{-1} is continuous: Let $(v_n)_{n \in \mathbb{N}}$ be a Cauchy-sequence of endomorphisms of \mathbf{B} . Since $(\text{End}(\mathbf{B}), d_{\bar{b}})$ is complete, $(v_n)_{n \in \mathbb{N}}$ is convergent — say to $v \in \text{End}(\mathbf{B})$.

Let $(\mathcal{U}, (f_U)_{U \in \mathcal{U}})$ be a strong gate covering of $\text{End}(\mathbf{B})$. Then there exists a $U \in \mathcal{U}$ and an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $v_n \in U$. Without loss of generality, assume that $n_0 = 0$. By the definition of strong gate coverings there exist Cauchy-sequences $(\kappa_n)_{n \in \mathbb{N}}$ and $(\iota_n)_{n \in \mathbb{N}}$ in $\overline{\text{Aut}(\mathbf{B})}$, such that $v_n = \kappa_n \circ f_U \circ \iota_n$, for all $n \in \mathbb{N}$. In particular, with

$$\kappa = \lim_{n \rightarrow \infty} \kappa_n \text{ and } \iota = \lim_{n \rightarrow \infty} \iota_n,$$

we have $v = \kappa \circ f_U \circ \iota$. Because g^{-1} is continuous, and since $g^{-1} = (h^{-1})|_{\overline{\text{Aut}(\mathbf{B})}}$, we have

$$\lim_{n \rightarrow \infty} h^{-1}(\kappa_n) = h^{-1}(\kappa) \text{ and } \lim_{n \rightarrow \infty} h^{-1}(\iota_n) = h^{-1}(\iota).$$

Now, since h^{-1} is a monoid-isomorphism, we have

$$h^{-1}(v_n) = h^{-1}(\kappa_n) \circ h^{-1}(f_U) \circ h^{-1}(\iota_n).$$

Thus, since the composition of functions is continuous, we have that the sequence $(h^{-1}(v_n))_{n \in \mathbb{N}}$ converges to $h^{-1}(v)$. From this, it follows that h^{-1} is continuous. \square

In the following we are going to adapt [5, Proposition 27] to the case of transformation monoids. In order to do so, we have to make a few preparations:

Definition 4.8. Let A be a countably infinite set and let $\mathfrak{M} \leq \mathfrak{T}_A$. For $a, b \in A$ define $a \preceq b$ if there exists some $h \in \mathfrak{M}$, such that $h(b) = a$. Let \sim be the closure of (\preceq) to an equivalence relation on A . Then the equivalence classes of \sim will be called *weak orbits* of \mathfrak{M} on A .

Lemma 4.9. Let A and B be sets, let $\varrho \subseteq A^2$ be a relation, and let $f: A \rightarrow B$ be a function, such that $\varrho \subseteq \ker f$. Then the closure ϱ^{eq} of ϱ to an equivalence relation is contained in $\ker f$, too.

Proof. This follows from the fact that the operator $-^{\text{eq}}$ is monotonous and idempotent. In particular we have

$$\varrho^{\text{eq}} \subseteq (\ker f)^{\text{eq}} = \ker f. \quad \square$$

Proposition 4.10. Let \mathbf{A} and \mathbf{B} be structures such that $\text{End}(\mathbf{A})$ contains all constant functions and such that $\text{End}(\mathbf{B})$ has only finitely many weak orbits on B . Then every monoid-isomorphism from $\text{End}(\mathbf{A})$ to $\text{End}(\mathbf{B})$ is open.

Proof. Let $h: \text{End}(\mathbf{A}) \rightarrow \text{End}(\mathbf{B})$ be a monoid-homomorphism. Let $a, b \in A$, and let $U = \{f \in \text{End}(\mathbf{A}) \mid f(a) = b\}$. For $d \in A$ denote by c_d the constant endomorphism of \mathbf{A} that maps everything to d . Then we have $U = \{f \in \text{End}(\mathbf{A}) \mid c_b = f \circ c_a\}$. Since h is a monoid-isomorphism, we have that $h(U) = \{g \in \text{End}(\mathbf{B}) \mid h(c_b) = g \circ h(c_a)\}$.

Note that c_a and c_b are left-zeros in $\text{End}(\mathbf{A})$. Thus, since h is a monoid-isomorphism, we have that $h(c_a)$ and $h(c_b)$ are left-zeros in $\text{End}(\mathbf{B})$. It follows that $h(c_a)$ and $h(c_b)$ are constant on weak orbits of $\text{End}(\mathbf{B})$ on B . Indeed, let $x \in B$, $g \in \text{End}(\mathbf{B})$, and let $y := g(x)$. Then $h(c_a) = h(c_a) \circ g$. Hence $h(c_a)(x) = h(c_a)(g(x)) = h(c_a)(y)$. In other words, $(\preceq) \subseteq \ker(h(c_a))$. Hence, by Lemma 4.9,

$$(\sim) = (\preceq)^{\text{eq}} \subseteq \ker(h(c_a)),$$

and the claim follows.

Let $\{o_1, \dots, o_k\}$ be a transversal of the weak orbits of $\text{End}(\mathbf{B})$ on B . Then we have for every $g \in \text{End}(\mathbf{B})$ that $h(c_b) = g \circ h(c_a)$ if and only if $h(c_b)(o_i) = g(h(c_a)(o_i))$, for all $i \in \{1, \dots, k\}$. In other words, with $a_i = h(c_a)(o_i)$ and $b_i = h(c_b)(o_i)$ ($i = 1, \dots, k$), we have

$$h(U) = \{g \in \text{End}(\mathbf{B}) \mid g(a_i) = b_i, i = 1, \dots, k\}.$$

Thus $h(U)$ is a finite intersection of basic open sets in $\text{End}(\mathbf{B})$. Consequently, $h(U)$ is open. \square

Remark. Note that for every transformation monoid $\mathfrak{M} \leq \mathfrak{T}_B$ we have that if the group \mathfrak{G} of units in \mathfrak{M} is oligomorphic, then \mathfrak{M} has only finitely many weak orbits on B . On the other hand, the monoid of non-expansive selfmaps of the rational Urysohn-space has just one weak

orbit but its automorphism group is not oligomorphic. Thus we have that the class of countable structures whose endomorphism monoid has only finitely many weak orbits properly contains the class of ω -categorical structures.

Theorem 4.11. *Let \mathbf{B} be a countable structure, such that $\text{End}(\mathbf{B})$ has only finitely many weak orbits on B , and let $h: \text{End}(\mathbb{Q}, \leq) \rightarrow \text{End}(\mathbf{B})$ be a monoid-isomorphism. Then h is a homeomorphism.*

Proof. Clearly, every constant function on \mathbb{Q} is an endomorphism of (\mathbb{Q}, \leq) . Thus, by Proposition 4.10, h is open.

It was shown by Kubiś [22, Proposition 3.23] that the class of finite linear orders has the AEP. It is known (cf. [28, 7]) that (\mathbb{Q}, \leq) is homomorphism-homogeneous. Thus, the class of finite linear orders has the HAP. Thus, by Proposition 2.5, (\mathbb{Q}, \leq) has a universal homogeneous endomorphism (this follows also from an earlier result [29, Proposition 4.7]).

Now, by Proposition 3.4, $\text{End}(\mathbb{Q}, \leq)$ has a strong gate covering. Finally, by Proposition 4.6, h is continuous. Altogether we have that h is a homeomorphism. \square

Corollary 4.12. *The endomorphism monoid $\text{End}(\mathbb{Q}, \leq)$ has automatic homeomorphicity with respect to the class of countable posets.*

Proof. Let $\mathbf{B} = (B, \leq)$ be a countable poset. Then every constant function on B is an endomorphism of \mathbf{B} hence $\text{End}(\mathbf{B})$ has just one weak orbit. Thus, by Theorem 4.11, every isomorphism from $\text{End}(\mathbb{Q}, \leq)$ to $\text{End}(\mathbf{B})$ is a homeomorphism. \square

Theorem 4.13. *Let \mathbf{B} be a countable structure, such that $\text{End}(\mathbf{B})$ has only finitely many weak orbits on B , and let h be a monoid isomorphism from the monoid of non-expansive selfmaps of the rational Urysohn space \mathbb{U}_0 to $\text{End}(\mathbf{B})$. Then h is a homeomorphism.*

Proof. Clearly, all constant functions on \mathbb{U}_0 are non-expansive. Thus, by Proposition 4.10, h is open.

The class of finite metric spaces has the AEP (cf. Example 2.12). It was shown by Dolinka in [12, Lemma 3.5] that the class of finite metric spaces has the HAP. Thus, by Proposition 2.5, \mathbb{U}_0 has a universal homogeneous endomorphism.

By Proposition 3.4, $\text{End}(\mathbb{U}_0)$ has a strong gate covering. Thus, by Proposition 4.6, h is continuous.

Altogether we have that h is a homeomorphism. \square

Corollary 4.14. *The monoid of non-expansive selfmaps of the rational Urysohn space has automatic homeomorphicity with respect to the class of countable metric spaces.*

Proof. Let \mathbf{M} be a countable metric space. Then every constant function on \mathbf{M} is a non-expansive selfmap of \mathbf{M} . Thus, the monoid of

non-expansive selfmaps of \mathbb{M} has just one weak orbit. Thus, by Theorem 4.13, every isomorphism between $\text{End}(\mathbb{U}_0)$ and $\text{End}(\mathbb{M})$ is a homeomorphism. \square

Recall that by (\mathbb{P}, \leq) is denoted the countable universal homogeneous partially ordered set (a.k.a. countable generic poset, or countable random poset).

Theorem 4.15. *Let \mathbf{B} be a countable structure, such that $\text{End}(\mathbf{B})$ has only finitely many weak orbits on B , and let $h: \text{End}(\mathbb{P}, \leq) \rightarrow \text{End}(\mathbf{B})$ be a monoid-isomorphism. Then h is a homeomorphism.*

Proof. Clearly, all constant functions are endomorphisms of (\mathbb{P}, \leq) . Thus, by Proposition 4.10, h is open.

The class of finite posets has the strict AP. Hence, it has the AEP. It was shown by Dolinka in [12, Example 3.4] that the class of finite posets has the HAP. Thus, by Proposition 2.5, (\mathbb{P}, \leq) has a universal homogeneous endomorphism.

By Proposition 3.4, $\text{End}(\mathbb{P}, \leq)$ has a strong gate covering. Thus, by Proposition 4.6, h is continuous.

Altogether we have that h is a homeomorphism. \square

5. CONCLUDING REMARKS

We conclude this paper with some open problems:

In [4] it was shown, that two positive existentially bi-interpretable ω -categorical structures have topologically isomorphic endomorphism monoids. Moreover, if two non-contractable ω -categorical structures have topologically isomorphic endomorphism monoids, then they are positive existentially bi-interpretable.

Unfortunately, we can not use this nice result to show reconstruction up to positive existential bi-interpretability, because our approach to show automatic homeomorphicity crucially depends on Proposition 4.6. In particular, all structures considered by us are contractable. We ask:

Problem. Is the rational Urysohn-space determined up to positive existential bi-interpretability by its endomorphism monoid, among all countable metric spaces?

Problem. Is (\mathbb{Q}, \leq) determined up to positive existential bi-interpretability by its endomorphism monoid, among all countable posets (chains)?

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INSTITUTE OF ALGEBRA, TU DRESDEN

E-mail address: `cpech@freenet.de`

URL: `http://www.math.tu-dresden.de/~pech`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOVI SAD

Current address: Institute of Algebra, TU Dresden

E-mail address: `maja@dmi.uns.ac.rs`, `Maja.Pech@tu-dresden.de`

URL: `http://sites.dmi.rs/personal/pechm/`